Partial Differential Equations

(Semester II; Academic Year 2024-25)

Indian Statistical Institute, Bangalore

Supplementary Examination

Duration: 3 hrs Maximum Marks: 50

1. Consider the IVP $u_y = u_x^3$, $u(x,0) = 2x^{\frac{3}{2}}$.
   (a) Discuss the existence and uniqueness of the IVP. (5)
   (b) Solve the IVP. (5)

2. Let Ω be a domain in $\mathbb{R}^2$ symmetric about the x-axis and let $\Omega^+ = \{(x,y) : y > 0\}$ be the upper part. Assume $u \in C(\overline{\Omega^+})$ is harmonic in Ω^+ with $u = 0$ on $\partial\Omega^+ \cap \{y = 0\}$. Define (10)
$$v(x,y) = \begin{cases} u(x,y), & y \geq 0, (x,y) \in \Omega, \\ -u(x,-y), & y < 0, (x,y) \in \Omega. \end{cases}$$
Show that v is harmonic.

3. Let ϕ be fundamental solution of heat equation given by:
$$\phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0, \\ 0, & x \in \mathbb{R}^n, t = 0. \end{cases}$$
   (a) Show that (5)
$$\lim_{(x,t)\to(x_0,0)} \phi(x,t) = 0, \quad \text{for } x_0 \neq 0.$$
   (b) Show that (5)
$$\int_{\mathbb{R}^n} \phi(x,t)dx = 1 \quad \forall t > 0.$$

4. Solve (10)
$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x > 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \\ u_x(0,t) = h(t). \end{cases}$$

5. Consider the wave equation: (10)
$$u_{tt} - \Delta u = 0, \quad \text{in } \mathbb{R}^n \times (0,\infty),$$
with initial data:
$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$
Verify that the solution is given by:
$$u(x,t) = u_\phi(x,t) + \int_0^t u_\psi(x,s)ds,$$
and also by:
$$u(x,t) = v_\psi(x,t) + \frac{\partial}{\partial t} v_\phi(x,t).$$

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